

ESSENTIAL DIMENSION, STABLE COHOMOLOGICAL DIMENSION, AND STABLE COHOMOLOGY OF FINITE HEISENBERG GROUPS

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ABSTRACT. We compare the notions of essential dimension and stable cohomological dimension of a finite group G , prove that the latter is bounded by the length of any normal series with cyclic quotients for G , and show that, however, this bound is not sharp by showing that the stable cohomological dimension of the finite Heisenberg groups H_p , p any prime, is equal to two.

1. INTRODUCTION

Let G be a finite group and F a finite G -module. The ground field will be the field of complex numbers unless otherwise specified for all objects below defined over a field. Then we define the *stable cohomology* $H_s^*(G, F)$, following [Bogo93], compare also the survey article [Bogo05], as a quotient of the usual group cohomology $H^*(G, F)$ by the ideal I_{unstable} of unstable classes which is the kernel of the map

$$H^*(G, F) \rightarrow \lim_U H^*(U/G, \tilde{F})$$

where U runs over (smaller and smaller) G -invariant nonempty Zariski open subset of a generically free (complex) linear G -representation V , contained in the open subset V^L of V on which the G -action has trivial stabilizers; \tilde{F} denotes the local system of abelian groups induced by F on (by slight abuse of notation) any of the U/G ; the kernel I_{unstable} does not depend on which V we choose.

Alternatively, if BGal_V denotes the absolute Galois group of $\mathbb{C}(V)^G$, we can say that $H_s^*(G, F)$ is equal to $H^*(G, F)$ modulo the kernel of the map

$$H^*(\text{BG}, \tilde{F}) \rightarrow H^*(\text{BGal}_V, \tilde{F})$$

derived from the natural map $f_V : \text{BGal}_V \rightarrow \text{BG}$. Both spaces BGal_V and BG carry natural topologies, and in general terms one could say

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that what the present article is about is to get a feeling for the homotopy type, call it Ω_G , of the image of f_V (again Ω_G is independent of V). Eventually, it would be highly desirable to have an explicit topological construction for Ω_G , starting from group-theoretic or combinatorial data associated to G and its representation theory. This is related to the problem to construct for any given finite group G an explicit discrete group Γ_G with a homomorphism $\Gamma_G \rightarrow G$ such that I_{unstable} coincides with the kernel of the map $H^*(G, F) \rightarrow H^*(\Gamma_G, F)$. Such a Γ_G exists, it suffices to take the fundamental group of a sufficiently small Zariski open U/G in V^L/G in the notation above, where U/G is an Eilenberg-McLane space $K(\Gamma_G, 1)$, but there is no constructive method to obtain U or, phrased differently, to tell how many divisors we have to remove to obtain stabilization.

Hence here we study some first dimension-theoretic properties of Ω_G in concrete examples. More precisely, the structure and roadmap of the paper is as follows: in Section 2 we give a definition for the basic notion of stable cohomological dimension, capturing the dimension theory of Ω_G from a cohomological point of view. We also discuss the related notion of essential dimension for G which captures the dimension theoretic properties of the class of free G -varieties V^L can map equivariantly to. We compare the two notions. Actually, in the case of a finite p -group, Merkurjev showed that the essential dimension is equal to the dimension of the smallest faithful representation of that group [KM], but the stable cohomological dimension is usually much smaller, in fact in Section 3 we prove that the stable cohomological dimension of a finite p -group G is bounded by the length of any normal series for G with cyclic quotients. In the subsequent sections we will show that the stable cohomological dimension is frequently even smaller than this bound.

In Section 4 we study stable cohomology of the finite Heisenberg group H_3 via a geometric construction taking its starting point from the classical Hesse pencil. In particular, we determine the stable cohomology of H_3 with constant coefficients of H_3 completely in this geometric way. Finally, in Section 5 we show that the stable cohomological dimension of the finite Heisenberg groups H_p , p any prime, is two, showing that the bound obtained in Section 3 is not sharp. We conclude with a conjecture that the top degree of nonvanishing stable cohomology with arbitrary coefficients is the same as that using only constant coefficients.

2. DEFINITIONS AND FIRST PROPERTIES

The notion of essential dimension was defined in [Reich]. The latter encapsulates only one but an important aspect of the behavior of the so called stable cohomology of groups which was introduced and studied in [Bogo88], [Bogo93], [BPT] among others.

Let G be an algebraic group. The word variety below will not imply that the object is irreducible. If this is intended, we will say irreducible variety.

- Definition 2.1.** (1) A *primitive* G -variety is a variety X with an action of G that permutes the components of X transitively. The G -action is called *generically free* if there is an open subset $U \subset X$, intersecting each irreducible component nontrivially, on which the G -action is free.
- (2) For a generically free G -variety one defines a G -compression as a dominant G -equivariant rational map $X \dashrightarrow Y$, where Y is another generically free G -variety.
- (4) The *essential dimension* $\text{ed}(X)$ of a primitive generically free G -variety X is the smallest possible value of $\dim(Y/G)$ where $X \dashrightarrow Y$ is a G -compression. The *essential dimension of G* , denoted $\text{ed}(G)$, is the essential dimension of a generically free linear G -representation.

The essential dimension of G is also equal to the smallest integer d such that any G -torsor on a function field over \mathbb{C} can be induced from a generically free algebraic G -scheme of dimension not exceeding d , see [GMS].

Let us introduce the second set of notions around which this paper is centered, going back to [Bogo93].

- Definition 2.2.** (1) Let X be an irreducible variety. The stable cohomological dimension of X , denoted $\text{scd}(X)$, is the maximum integer n such that there exists a locally constant sheaf of finite abelian groups F on X with $H_s^n(X, F) \neq 0$.
- (2) Let G be an algebraic group. Then the stable cohomological dimension $\text{scd}(G)$ is defined as the maximum integer n such that there exists a finite G -module M with $H_s^n(G, M) \neq 0$.

Remark 2.3. The numbers $\text{scd}(X)$ and $\text{scd}(G)$ are clearly finite: in the first case, it is bounded by $\dim X$, in the second by $\dim V/G$ where V is a generically free G -representation.

First we state some immediate consequences of the definitions.

Proposition 2.4. (1) *If G is a finite group and $H \subset G$ a subgroup, then $\mathrm{scd}(G) \geq \mathrm{scd}(H)$.*

(2) *For a finite group G one has*

$$\mathrm{scd}(G) = \max_p \{\mathrm{scd}(\mathrm{Syl}_p(G))\}.$$

(3) *One always has the inequality $\mathrm{scd}(G) \leq \mathrm{ed}(G)$.*

Proof. One sees (1) from the equality

$$H^d(H, N) = H^d(G, \mathrm{Ind}_H^G(N))$$

for a finite module N over H , and the fact that stable classes in the cohomology of the left hand side give stable classes for the cohomology of G on the right hand side (restrict to H to see this).

For (2) notice that we have a decomposition of cohomology groups $H_s^*(G, M) = \bigoplus_p H_s^*(G, M_{(p)})$ where $M_{(p)}$ is the p -primary component of the finite G -module M , and an injection

$$H_s^*(G, M_{(p)}) \hookrightarrow \bigoplus_p H_s^*(\mathrm{Syl}_p(G), M_{(p)}).$$

This shows that $\mathrm{scd}(G) \leq \max_p \{\mathrm{scd}(\mathrm{Syl}_p(G))\}$. On the other hand, by (1), $\mathrm{scd}(G) \geq \max_p \{\mathrm{scd}(\mathrm{Syl}_p(G))\}$ holds as well.

For (3), let V be a generically free G -representation. Let $V \dashrightarrow Y$ be a G -compression inducing a dominant map $V/G \dashrightarrow Y/G$ with $\dim Y/G \leq \mathrm{ed}(G)$. The (topological) map from V^0/G to the classifying space BG factors:

$$V^0/G \rightarrow Y^0/G \rightarrow BG$$

where V^0 and Y^0 are open subsets where the action of G is free. Hence the stable cohomology of G vanishes in degrees larger than $\dim Y/G$. \square

Remark 2.5. It was shown in [KM] that the essential dimension of a finite p -groups is equal to the minimal dimension of a faithful linear representation of the group. The essential dimension of a finite abelian group equals its rank (see for example [BR], Thm. 6.1), hence is equal to the stable cohomological dimension. The known values for $\mathrm{ed}(\mathfrak{S}_n)$, the essential dimension of the symmetric group, can be found in [M12], Thm. 3.22. In particular, $\mathrm{ed}(\mathfrak{S}_7) = 4$ whereas $\mathrm{scd}(\mathfrak{S}_7) = 3$; the latter can be seen because the stable cohomological dimension of $\mathbb{Z}/2 \wr \mathbb{Z}/2 \times \mathbb{Z}/2 = \mathrm{Syl}_2(\mathfrak{S}_7)$ is 3 (cf. [B-B12] where the stable cohomology of wreath products is determined). It shows that one can have strict inequality $\mathrm{scd}(G) < \mathrm{ed}(G)$.

Example 2.6. For an iterated wreath product $\mathbb{Z}/p \wr \mathbb{Z}/p \wr \cdots \wr \mathbb{Z}/p$ (n factors) of cyclic groups of prime order p , the essential dimension and stable cohomological dimension coincide and are equal to p^{n-1} . This is immediate from [KM] (one has a faithful representation of dimension p^{n-1}) and [B-B12].

3. BOUNDS ON STABLE COHOMOLOGICAL DIMENSION VIA NORMAL SERIES

In this section, we bound $\text{scd}(G)$ for a finite p -group G in terms of the length of a normal series for G with cyclic quotients.

Theorem 3.1. *Let G be a finite p -group and let*

$$\{1\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

be a normal series for G , thus G_i is normal in the next group G_{i+1} . Suppose that the quotients G_{i+1}/G_i are cyclic. Then $\text{scd}(G) \leq n$.

Proof. We use induction on n , the case $n = 1$ being clear. Thus we can suppose $H \triangleleft G$ is a normal subgroup with $G/H \simeq \mathbb{Z}/p^k$ cyclic, and we will show that $\text{scd}(G) \leq \text{scd}(H) + 1$.

Take a faithful representation V of G , and let V^L be an open subvariety of V which is a $K(\pi, 1)$ -space and generically free for the G -action. Let $\chi : G \rightarrow \mathbb{Z}/p^k$ be a surjection with kernel H , and let \mathbb{C}_χ be a one dimensional representation where G acts via the character χ . The representation $V \oplus \mathbb{C}_\chi$ contains an open subvariety $U := V^L \times (\mathbb{C}_\chi - \{0\})$ which is a $K(\pi, 1)$ and on which G acts freely. We obtain a natural homomorphism $\pi_1(U/G) \rightarrow G$ which gives a partial stabilization: the kernel of the map $H^*(G, F) \rightarrow H_s^*(G, F)$ contains the kernel of the map $H^*(G, F) \rightarrow H^*(\pi_1(U/G), F)$ for an arbitrary finite coefficient module F . Let $G_{\mathbb{Z}}$ be the fiber product

$$\begin{array}{ccc} G_{\mathbb{Z}} & \xrightarrow{\pi_{\mathbb{Z}}} & \mathbb{Z} \\ \downarrow & & \downarrow \\ G & \xrightarrow{\chi} & \mathbb{Z}/p^k. \end{array}$$

Then there is a commutative triangle

$$\begin{array}{ccc} \pi_1(U/G) & \longrightarrow & G_{\mathbb{Z}} \\ & \searrow & \downarrow \\ & & G \end{array}$$

hence the kernel of $H^*(G, F) \rightarrow H^*(G_{\mathbb{Z}}, F)$ is also killed under stabilization. Consider the spectral sequence arising from the projection

$G_{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}$ with kernel H . Since $H^i(\mathbb{Z}, L) = 0$ for $i > 1$ and any \mathbb{Z} -module L , the spectral sequence splits into short exact sequences

$$0 \longrightarrow H^1(\mathbb{Z}, H^{i-1}(H, F)) \longrightarrow H^i(G_{\mathbb{Z}}, F) \longrightarrow H^0(\mathbb{Z}, H^i(H, F)) \longrightarrow 0$$

and stabilization gives a map between exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(\mathbb{Z}, H^{i-1}(H, F)) & \longrightarrow & H^i(G_{\mathbb{Z}}, F) & \longrightarrow & H^0(\mathbb{Z}, H^i(H, F)) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(\mathbb{Z}, H^{i-1}(H_s, F)) & \longrightarrow & H^i(G_{\mathbb{Z}}, F) & \longrightarrow & H^0(\mathbb{Z}, H^i(H_s, F)) \longrightarrow 0 \end{array}$$

where H_s is a group stabilizing the cohomology of H . Hence, the image of $H^i(G_{\mathbb{Z}}, F)$ is zero in the stable cohomology of the fibration $U \rightarrow (\mathbb{C}_X - \{0\})$ for $i \geq \text{scd}(H) + 2$. Thus $\text{scd}(G) \leq \text{scd}(H) + 1$ as claimed. \square

4. THE HEISENBERG GROUP H_3 AND THE HESSE PENCIL

Let H_3 be the Heisenberg group over the finite field \mathbb{F}_3 sitting in an extension

$$0 \rightarrow \mathbb{Z}/3 \rightarrow H_3 \rightarrow \mathbb{Z}/3 \oplus \mathbb{Z}/3 \rightarrow 0$$

with the standard generators x and y in the two copies of the quotient $\mathbb{Z}/3 \oplus \mathbb{Z}/3$ satisfying $x^3 = y^3 = 1$, $x^{-1}y^{-1}xy = z$ where z is a generator of the central $\mathbb{Z}/3$. If E is a smooth elliptic curve and \mathcal{L} a degree 3 line bundle on E , then H_3 acts on $V^* = H^0(E, \mathcal{L})$ via the standard Schrödinger representation, embedding E into \mathbb{P}^2 as a curve of the Hesse pencil

$$\lambda(z_1^3 + z_2^3 + z_3^3) + \mu z_1 z_2 z_3 = 0,$$

where the coordinates $z_1, z_2, z_3 \in H^0(E, \mathcal{L})$ are chosen such that $x \in \mathbb{Z}/3$ acts as $z_i \mapsto \omega^i z_i$ with ω a primitive cube root of unity, and y acts as $z_i \mapsto z_{i-1}$ with lower indices interpreted mod. 3. For much of the geometry associated to it a good reference is [ArtDolg].

The nine base points of the pencil, call the set of those Σ , are the inflection points of any curve in the pencil and located at

$$(1 : -\varrho : 0), \quad (0 : 1 : -\varrho), \quad (-\varrho : 0 : 1), \quad \varrho^3 = 1.$$

The Hesse pencil gives a rational map $\mathbb{P}^2 = \mathbb{P}(V) \dashrightarrow \mathbb{P}^1$, $(z_1 : z_2 : z_3) \mapsto (z_1 z_2 z_3, z_1^3 + z_2^3 + z_3^3)$, undefined at the nine base points, whence after blowing those up we get a regular map

$$\varphi : S(3) \rightarrow \mathbb{P}^1$$

where $S(3)$ is a rational surface having the structure of minimal elliptic surface via φ whose fibers are the members of the Hesse pencil. $S(3)$ is a so-called *elliptic modular surface of level 3*. If $\{b_1, \dots, b_k\} \subset \mathbb{P}^1$

are the points corresponding to singular fibers of $S(3)$, $(\mathbb{P}^1)^0$, $S(3)^0$ the corresponding open complements, $\varphi^0 : S(3)^0 \rightarrow (\mathbb{P}^1)^0$ the induced map on the complement of the singular fibers, then φ^0 is topologically a smooth oriented fibration with fiber $S^1 \times S^1$ with a section over \mathbb{P}^1 . It is classified by the homomorphism $\pi_1((\mathbb{P}^1)^0) \rightarrow \mathrm{SL}_2(\mathbb{Z})$ (and the topology of the base). Its image is the monodromy group of the fibration. In this case it is the modular group

$$\Gamma(3) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{3} \right\}.$$

There is a commutative diagram of fibrations

$$\begin{array}{ccc} \mathbb{C}^* \simeq \mathrm{B}\mathbb{Z} & \xrightarrow{\quad\quad\quad} & \mathrm{B}(\mathbb{Z}/3) \\ \downarrow & & \downarrow \\ (V \setminus \pi^{-1}(\Sigma))/H_3 & \xrightarrow{\quad\quad\quad} & \mathrm{B}H_3 \\ \downarrow \pi & & \downarrow \\ (\mathbb{P}(V) \setminus \Sigma)/(\mathbb{Z}/3)^2 \simeq (S(3) \setminus (9 \text{ sections})) / (\mathbb{Z}/3)^2 & \xrightarrow{\quad\quad\quad} & \mathrm{B}(\mathbb{Z}/3 \oplus \mathbb{Z}/3) \end{array}$$

and, correspondingly, we have two spectral sequences of fibrations

$$\begin{aligned} E_2^{pq} &= H^p(\mathbb{Z}/3 \oplus \mathbb{Z}/3, H^q(\mathbb{Z}/3)) \implies H^{p+q}(H_3), \\ E_2^{pq} &= H^p((S(3) \setminus (9 \text{ sections})) / (\mathbb{Z}/3)^2, \mathcal{H}^q(\mathbb{C}^*)) \implies H^{p+q}((V \setminus \pi^{-1}(\Sigma))/H_3) \end{aligned}$$

(all cohomology taken with some constant coefficients F here, and we write $\mathcal{H}^q(\mathbb{C}^*)$ to indicate that this is a local system on the base), connected by a morphism of spectral sequences induced by the diagram. We are interested in the image of the cohomology $H^*(H_3)$ in the direct limit of the cohomologies of complements of divisors in V/H_3 (i.e., we are interested in $H_s^*(H_3)$).

Lemma 4.1. *For any constant coefficients F , the image of the cohomology $H^2(\mathbb{Z}/3 \oplus \mathbb{Z}/3, F)$ in the stable cohomology*

$$H_s^2(S(3)/(\mathbb{Z}/3 \oplus \mathbb{Z}/3), F)$$

is zero.

Proof. By e.g. [ArtDolg], Prop. 5.1, the quotient surface $S(3)/(\mathbb{Z}/3 \oplus \mathbb{Z}/3)$ has four singular points s_1, \dots, s_4 of type A_2 , given by the orbits of the vertices of the four singular members of the Hesse pencil, which are four triangles. Analytically locally, they are of type $\mathbb{C}^2/(\mathbb{Z}/3)$. Outside

of the vertices of the triangles, the action is free. Thus the map

$$H^2(\mathbb{Z}/3 \oplus \mathbb{Z}/3, F) \rightarrow H_s^2(S(3)/(\mathbb{Z}/3 \oplus \mathbb{Z}/3), F)$$

factors over $S(3)/((\mathbb{Z}/3 \oplus \mathbb{Z}/3) - \{s_1, \dots, s_4\})$ and also, denoting by \tilde{S} the minimal resolution of $S(3)/(\mathbb{Z}/3 \oplus \mathbb{Z}/3)$, D_1^\pm, \dots, D_4^\pm the divisors lying over s_1, \dots, s_4 , the map factors over $H^2(\tilde{S} - \{D_1^\pm, \dots, D_4^\pm\}, F)$. Now we claim that the image in

$$H^2(\text{Gal}(\mathbb{C}(\tilde{S})), F)$$

of every element coming from $H^2(\mathbb{Z}/3 \oplus \mathbb{Z}/3, F)$ has zero residue with respect to any of the divisorial valuations associated to the D_i^\pm : in fact, the image of the decomposition groups associated to these valuations in $\mathbb{Z}/3 \oplus \mathbb{Z}/3$ are of type $\mathbb{Z}/3$, and the residue maps factor over the stable cohomology of these groups $\mathbb{Z}/3$. Hence any element coming from $H^2(\mathbb{Z}/3 \oplus \mathbb{Z}/3, F)$ extends from the open part $\tilde{S} - \{D_1^\pm, \dots, D_4^\pm\}$ through the generic point of any of the divisors D_i^\pm and hence to all of \tilde{S} . Since \tilde{S} is a rational surface, its second cohomology consists only of classes of algebraic line bundles. All of these are killed under the stabilization morphism. \square

Proposition 4.2. *The stable cohomology of the group H_3 in degree 3*

$$H_s^3(H_3, \mathbb{Z}/3)$$

is zero. The ring $H_s^(H_3, \mathbb{Z}/3)$ can be described by saying that it is generated by elements y, y' of degree 1, Y, Y' of degree 2, and all products between any two of these are zero.*

Proof. We restrict the left hand fibration in the diagram

$$\begin{array}{ccc} \mathbb{C}^* \simeq B\mathbb{Z} & \xrightarrow{\quad\quad\quad} & B(\mathbb{Z}/3) \\ \downarrow & & \downarrow \\ (V \setminus \pi^{-1}(\Sigma))/H_3 & \xrightarrow{\quad\quad\quad} & BH_3 \\ \downarrow \pi & & \downarrow \\ (\mathbb{P}(V) \setminus \Sigma)/(\mathbb{Z}/3)^2 \simeq (S(3) \setminus (9 \text{ sections}))/(\mathbb{Z}/3)^2 & \xrightarrow{\quad\quad\quad} & B(\mathbb{Z}/3 \oplus \mathbb{Z}/3) \end{array}$$

to an open subset $U \subset (\mathbb{P}(V) \setminus \Sigma)/(\mathbb{Z}/3)^2$ chosen so small that

- (a) The cohomology of \tilde{S} is stabilized on it.
- (b) It is a $K(\Gamma, 1)$.

Then we look at the induced map of spectral sequences

$$\begin{aligned} E_2^{pq} &= H^p(\mathbb{Z}/3 \oplus \mathbb{Z}/3, H^q(\mathbb{Z}/3, \mathbb{Z}/3)) \implies H^{p+q}(H_3, \mathbb{Z}/3), \\ E_2^{pq} &= H^p(\Gamma, H^q(\mathbb{Z}, \mathbb{Z}/3)) \implies H^{p+q}(\pi^{-1}(U), \mathbb{Z}/3) \end{aligned}$$

to find that the image of $H^3(H_3, \mathbb{Z}/3)$ in $H^3(\pi^{-1}(U), \mathbb{Z}/3)$ is zero; note that any element coming from $H^2(\mathbb{Z}/3 \oplus \mathbb{Z}/3, \mathbb{Z}/3)$ extends to \tilde{S} , i.e. the map to the cohomology of U factors through the cohomology of \tilde{S} . The remaining assertions follow from Theorem 7 of [Leary] and the fact that the Steenrod operations are trivial in stable cohomology. \square

Remark 4.3. This result agrees with what is stated in [TezYag], Prop. 10.2, though the argument here is different and maybe more elementary.

5. STABLE COHOMOLOGICAL DIMENSION OF ARBITRARY HEISENBERG GROUPS

Let H_p be the Heisenberg group H_p , p some prime number, which is a central extension

$$0 \rightarrow \mathbb{Z}/p \rightarrow H_p \rightarrow \mathbb{Z}/p \oplus \mathbb{Z}/p \rightarrow 0$$

defined by thinking of elements of H_p as given by matrices

$$M(a, b, c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c \in \mathbb{Z}/p$ and group composition given by matrix multiplication. Let us write temporarily $A = \mathbb{Z}/p$, $\hat{A} = \text{Hom}^{\text{gp}}(\mathbb{Z}/p, \mathbb{C}^*) \simeq \mathbb{Z}/p$, and the isomorphism $A \simeq \hat{A}$ is given by the assignment

$$A \ni k \mapsto \left(\mathbb{Z}/p \ni j \mapsto \chi_k(j) = e^{\frac{2\pi i}{p}kj} \in S^1 \subset \mathbb{C}^* \right) \in \hat{A}.$$

Let V be the p -dimensional complex vector space of complex valued functions on the set A . For a pair $(a, \chi) \in A \times \hat{A}$ one can define the Weyl operator $W_{(a, \chi)}$, a linear self-map from V to V , by

$$(W_{(a, \chi)}f)(u) = \chi(u)f(a + u), \quad f \in V, u \in A$$

and the standard p -dimensional Schrödinger representation ϱ of H_p on V by

$$\varrho(M(a, b, c)) = e^{\frac{2\pi i}{p}c} W_{(a, \chi_b)}.$$

Consider the action of H_p on $V \times T$ where $T = \mathbb{C}^* \times \mathbb{C}^*$ and H_p acts on T via the quotient $\Gamma := \mathbb{Z}/p \times \mathbb{Z}/p$, multiplying each coordinate by

a p -th root of unity. There is an induced action of the quotient Γ on $\mathbb{P}(V) \times T$ and a \mathbb{C}^* -fibration

$$(1) \quad f : (V - \{0\} \times T)/H_p \rightarrow (\mathbb{P}(V) \times T)/\Gamma.$$

Let $H \subset \mathbb{P}(V)$ be a general hyperplane and let \mathcal{H} be its Γ -orbit. Then \mathcal{H} is a generic hyperplane arrangement and $\Omega := \mathbb{P}(V) - \mathcal{H}$ has fundamental group \mathbb{Z}^{p^2-1} . Moreover,

$$(\Omega \times T)/\Gamma \subset (\mathbb{P}(V) \times T)/\Gamma$$

has a fundamental group Π which is an extension (as is seen by looking at the projection $(\Omega \times T)/\Gamma \rightarrow T/\Gamma$)

$$(2) \quad 0 \rightarrow \mathbb{Z}^{p^2-1} \rightarrow \Pi \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow 0$$

where \mathbb{Z}^{p^2-1} is a Γ -module (hence a $\mathbb{Z} \oplus \mathbb{Z}$ -module via the surjection $\mathbb{Z} \oplus \mathbb{Z} \twoheadrightarrow \mathbb{Z}/p \oplus \mathbb{Z}/p$ which is the product of the canonical projections in both factors), equal to

$$\mathbb{Z}[\Gamma]/\mathbb{Z} \left(\sum_{\gamma \in \Gamma} \gamma \right)$$

since the fundamental group of the fiber is generated by loops around the hyperplanes in \mathcal{H} and the sum of these is trivial.

Lemma 5.1. *The extension (2) is not a semi-direct product.*

Proof. Notice first, to get the right picture of the topological situation, that the class of the projective bundle

$$\omega : (\mathbb{P}(V) \times T)/\Gamma \rightarrow T/\Gamma$$

in the topological Brauer group of $T/\Gamma \simeq \mathbb{C}^* \times \mathbb{C}^*$ is trivial: indeed, the topological Brauer group of any good topological space X , say a finite CW-complex, in our case $\mathbb{C}^* \times \mathbb{C}^*$, which is homotopy equivalent to $S^1 \times S^1$, is nothing but $H_{\text{tors}}^3(X, \mathbb{Z})$, so this vanishes identically. Since we can find an $S^1 \times S^1$ in the complement of any divisor in T/Γ , the preceding projective bundle has a section Σ . However, we claim that there cannot be any such section Σ contained in the complement of the hyperplane arrangement $(\Omega \times T)/\Gamma$: indeed, the Picard group of $\mathbb{P}(V) - \mathcal{H}$ is torsion, hence the \mathbb{C}^* -fibration over $(\Omega \times T)/\Gamma$ corresponds to a torsion line bundle and would restrict trivially to Σ . Hence the fundamental group \mathbb{Z}^3 of this restricted \mathbb{C}^* -fibration cannot surject onto the nonabelian group H_p .

Note that if we do not remove \mathcal{H} , then we can find a Σ with a nontrivial \mathbb{C}^* -fibration over it and fundamental group a nontrivial extension of $\mathbb{Z} \oplus \mathbb{Z}$ by \mathbb{Z} , surjecting onto H_p . \square

We need some auxiliary results about stable cohomology of Γ with arbitrary coefficients to proceed.

Proposition 5.2. *Let M be a finite $\Gamma = \mathbb{Z}/p \oplus \mathbb{Z}/p$ -module annihilated by a power of p . Let $c := \sum_{\gamma \in \Gamma} \gamma \in \mathbb{Z}[\Gamma]$ and let $M_c \subset M$ be the submodule consisting of elements annihilated by c . Then the natural map*

$$H_s^2(\Gamma, M_c) \rightarrow H_s^2(\Gamma, M)$$

is surjective.

We first remark

Lemma 5.3. *Let F be a finite \mathbb{Z}/p module, $t := \sum_{g \in \mathbb{Z}/p} g \in \mathbb{Z}[\mathbb{Z}/p]$, and $F_t \subset F$ the submodule annihilated by t . Then the natural map*

$$H^1(\mathbb{Z}/p, F_t) \rightarrow H^1(\mathbb{Z}/p, F)$$

is surjective.

Proof. Recall that for any group G and G -module M , $H^1(G, M)$ can be interpreted as derivations $f : G \rightarrow M$, i.e. maps f from G to M satisfying

$$f(gh) = f(g) + gf(h) \quad \forall g, h \in G,$$

modulo principal derivations, which are those mapping $g \in G$ to $(1 - g)m$ for some fixed $m \in M$. Now a derivation $f : \mathbb{Z}/p \rightarrow F$ is determined by the image of a generator x of \mathbb{Z}/p , $f(x) = m$ say, but since $x^p = \text{id}$, not every $m \in F$ can occur: since

$$0 = f(x^p) = (1 + x + \cdots + x^{p-1})m = t \cdot m$$

only those m which lie in F_t can be potential images of x , but this is the only restriction. Hence

$$H^1(\mathbb{Z}/p, F) = F_t/[\mathbb{Z}/p, F].$$

The natural map $H^1(\mathbb{Z}/p, F_t) \rightarrow H^1(\mathbb{Z}/p, F)$ is nothing other than the map

$$F_t/[\mathbb{Z}/p, F_t] \rightarrow F_t/[\mathbb{Z}/p, F]$$

which is visibly surjective. □

Now we turn to the

Proof. (of Proposition 5.2) The stabilization morphism $H^2(\mathbb{Z}/p \oplus \mathbb{Z}/p, F) \rightarrow H_s^2(\mathbb{Z}/p \oplus \mathbb{Z}/p, F)$ factors over the natural map

$$H^2(\mathbb{Z}/p \oplus \mathbb{Z}/p, F) \rightarrow H^2(\mathbb{Z} \oplus \mathbb{Z}, F) = H^1(\mathbb{Z}, H^1(\mathbb{Z}, F))$$

where for the last isomorphism we use the Lyndon-Hochschild-Serre spectral sequence applied to the trivial extension $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow$

$\mathbb{Z} \rightarrow 0$. Denote by C_1, C_2 the two coordinate copies of \mathbb{Z}/p in $\Gamma = \mathbb{Z}/p \oplus \mathbb{Z}/p$.

Claim: The natural map

$$H^1(C_1, H^1(C_2, F_c)) \rightarrow H^1(C_1, H^1(C_2, F))$$

is surjective.

Let us denote by t_1 resp. t_2 the elements in $\mathbb{Z}[C_i]$, $i = 1$ resp. $= 2$, which are the sums of all the elements in the group. Then by Lemma 5.3 we have that $H^1(C_1, H^1(C_2, F))$ is equal to

$$H^1(C_1, F_{t_2}/[C_2, F]) = \{x \in F_{t_2}/[C_2, F] \mid t_1 \cdot x = 0\} / [C_1, F_{t_2}/[C_2, F]].$$

In other words, elements in this group are given by elements in F which are annihilated by t_2 and mapped by multiplication by t_1 into $[C_2, F]$, but considered modulo elements in a certain commutator submodule. But every element in F_{t_2} is automatically in F_c since

$$c = t_1 \cdot t_2 \in \mathbb{Z}[\Gamma],$$

hence the Claim follows.

Now $H^1(\mathbb{Z}/p, H^1(\mathbb{Z}/p, F))$ is in fact a subspace of $H^2(\mathbb{Z}/p \oplus \mathbb{Z}/p, F)$ (the assertion of course with F replaced by F_c) which surjects onto $H_s^2(\mathbb{Z}/p \oplus \mathbb{Z}/p, F)$: the fact that it is a subspace follows by looking at the Hochschild-Lyndon-Serre spectral sequence

$$E_2^{p,q} = H^p(C_1, H^q(C_2, F)) \implies H^2(\mathbb{Z}/p \oplus \mathbb{Z}/p, F)$$

and remarking that the differential

$$d_2 : H^1(C_1, H^1(C_2, F)) \rightarrow H^3(C_1, H^0(C_2, F))$$

is zero because the cohomology of the base $H^3(C_1, H^0(C_2, F))$ injects into the cohomology of the total space, since the group extension under consideration is trivial, hence has a section. The terms $H^2(C_1, H^0(C_2, F))$ and $H^0(C_1, H^2(C_2, F))$ of the E_2 -page are already mapped to zero under the morphism of spectral sequences to

$$E_2^{p,q} = H^p(\mathbb{Z}, H^q(\mathbb{Z}, F)) \implies H^2(\mathbb{Z} \oplus \mathbb{Z}, F)$$

and, as remarked above, the stabilization morphism factors over $H^2(\mathbb{Z} \oplus \mathbb{Z}, F)$. Hence $H^1(\mathbb{Z}/p, H^1(\mathbb{Z}/p, F))$ surjects onto $H_s^2(\mathbb{Z}/p \oplus \mathbb{Z}/p, F)$. \square

Definition 5.4. Let M be a module for Γ (or $\mathbb{Z}[\Gamma]$, which amounts to the same) which is finite, annihilated by some power of p , and annihilated by $c = \sum_{\gamma \in \Gamma} \gamma \in \mathbb{Z}[\Gamma]$. We call such an M a *basic module* for Γ .

Hence, if M is a basic Γ -module, then any map

$$\mathbb{Z}[\Gamma] \rightarrow M$$

onto a one generator submodule factors over $\mathbb{Z}[\Gamma]/(c)$. The next Lemma won't be needed for the derivation of the main result, but has some independent interest.

Lemma 5.5. *If M is a rank one $\mathbb{Z}[\Gamma]$ -module isomorphic to $(\mathbb{Z}/p)^N$ as an abelian group for some N and not free as a $\mathbb{Z}/p[\Gamma]$ -module, then*

$$\left(\sum_{\gamma \in \Gamma} \gamma \right) \cdot m$$

is trivial in M , hence M is basic.

Proof. We calculate in $\mathbb{Z}/p[\Gamma]$. Write $x := 1 - g_1, y := 1 - g_2 \in \mathbb{Z}/p[\Gamma]$, where g_1, g_2 are generators of the two coordinate copies of \mathbb{Z}/p in $\Gamma = \mathbb{Z}/p \times \mathbb{Z}/p$. Then every $z \in \mathbb{Z}/p[\Gamma]$ can be written as

$$z = \sum_{1 \leq i, j \leq p-1} a_{ij} x^i y^j, \quad a_{ij} \in \mathbb{Z}/p.$$

Since M is not free as a $\mathbb{Z}/p[\Gamma]$ -module, there will be a nonzero such z in the kernel of the surjection $\mathbb{Z}/p[\Gamma] \twoheadrightarrow M$ defined by the generator m . The kernel being a left-ideal (we consider M as a left $\mathbb{Z}/p[\Gamma]$ -module), we see that also $x^{p-1}y^{p-1}$ is in the kernel (for this, let $x^{i_0}y^{j_0}$ be the monomial occurring in z with nonzero coefficient which has lexicographically smallest (i, j) , and multiply z by $x^{p-1-i_0}y^{p-1-j_0}$, and note that $x^p = y^p = 0$). Now

$$x^{p-1}y^{p-1} = \left(\sum_{i=1}^{p-1} (-1)^i \binom{p-1}{i} g_1 \right) \left(\sum_{j=1}^{p-1} (-1)^j \binom{p-1}{j} g_2 \right)$$

and $(-1)^i \binom{p-1}{i} = 1$ in \mathbb{Z}/p , so $x^{p-1}y^{p-1} = \sum_{\gamma \in \Gamma} \gamma$. \square

Proposition 5.6. *If M is a basic Γ -module, then the image of $H^2(\Gamma, M)$ is trivial in $H^2((\Omega \times T)/\Gamma, \tilde{M})$ where \tilde{M} is the local system on $(\Omega \times T)/\Gamma$ induced by M .*

Proof. We have to show that given a nontrivial extension in $H^2(\Gamma, M)$:

$$0 \rightarrow M \rightarrow H \rightarrow \Gamma \rightarrow 0$$

(where by definition Γ acts on the abelian normal subgroup M of H via conjugation and this is the given action of Γ on M), this extension

becomes trivial when we pull it back to Π via the surjection $\Pi \twoheadrightarrow \mathbb{Z} \oplus \mathbb{Z} \twoheadrightarrow \mathbb{Z}/p \oplus \mathbb{Z}/p$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & H_{\Pi} & \longrightarrow & \Pi \longrightarrow 0 \\ & & \parallel & & \downarrow & \swarrow & \downarrow \\ 0 & \longrightarrow & M & \longrightarrow & H & \longrightarrow & \Gamma \longrightarrow 0 \end{array}$$

That the upper extension splits (i.e., is a semi-direct product) is equivalent to the existence of a dashed arrow as shown. In fact, given such an arrow, we get a section of the projection $H_{\Pi} \rightarrow \Pi$ since H_{Π} is a fiber product of H and Π over Γ ; and given a section, we get the required dashed arrow by composing with $H_{\Pi} \rightarrow H$.

We claim that we can form some commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^{p^2-1} & \longrightarrow & \Pi & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0 \\ & & \downarrow \pi & & \downarrow \varphi & & \downarrow \\ 0 & \longrightarrow & M & \longrightarrow & H & \longrightarrow & \mathbb{Z}/p \oplus \mathbb{Z}/p \longrightarrow 0 \end{array}$$

with the right hand vertical arrow $\mathbb{Z} \oplus \mathbb{Z} \twoheadrightarrow \mathbb{Z}/p \oplus \mathbb{Z}/p$ the product of the two canonical projections $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p$. In other words, we have to justify the existence of the dashed arrows.

To do this, suppose that M is annihilated by p^k ; then we know that every submodule of M generated by one element is the image of a map

$$\mathbb{Z}/(p^k)[\Gamma]/(c) \rightarrow M$$

since M is basic. The reduction map $\mathbb{Z}[G]/(c) \twoheadrightarrow \mathbb{Z}/(p^k)[G]/(c)$ induces a morphism of extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}[\Gamma]/(c) & \longrightarrow & \Pi & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathbb{Z}/(p^k)[\Gamma]/(c) & \longrightarrow & \Pi' & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0 \end{array}$$

and it suffices to define maps φ' , π' , making

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/(p^k)[\Gamma]/(c) & \longrightarrow & \Pi' & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0 \\ & & \downarrow \pi' & & \downarrow \varphi' & & \downarrow \\ 0 & \longrightarrow & M & \longrightarrow & H & \longrightarrow & \mathbb{Z}/p \oplus \mathbb{Z}/p \longrightarrow 0 \end{array}$$

commutative.

We want to define the map φ' in the following way: let $a = (1, 0), b = (0, 1) \in \mathbb{Z} \oplus \mathbb{Z}$, and $\alpha = (\bar{1}, \bar{0}), \beta = (\bar{0}, \bar{1}) \in \mathbb{Z}/p \oplus \mathbb{Z}/p = \Gamma$. Then we claim that we get a *well-defined homomorphism* φ' by the rule

$$\tilde{a} \mapsto \tilde{\alpha}, \quad \tilde{b} \mapsto \tilde{\beta}$$

where \tilde{a}, \tilde{b} are arbitrary lifts of a, b to Π' and $\tilde{\alpha}, \tilde{\beta}$ equally arbitrary lifts of α, β to H . In fact, we will show that then $[\tilde{a}, \tilde{b}]$ is a generator of the Γ -module $\mathbb{Z}/(p^k)[\Gamma]/(c)$ whence the map π' is the one mapping this generator to $[\tilde{\alpha}, \tilde{\beta}]$ which generates a rank 1 submodule of M . Then π' and with it also φ' will be well-defined since $\mathbb{Z}/(p^k)[\Gamma]/(c)$ can be mapped in a well-defined way into any basic module annihilated by p^k by assigning the image of a generator arbitrarily.

We now use that the extension (2) is not a semi-direct product by Lemma 5.1 to show that $\gamma := [\tilde{a}, \tilde{b}]$ is a generator of the Γ -module $F := \mathbb{Z}/(p^k)[\Gamma]/(c)$: in fact, by hypothesis, the class $\bar{\gamma}$ in $\bar{F} := F/pF$ maps to a generator of $\bar{F}/[\Gamma, \bar{F}] \simeq \mathbb{Z}/p$. Hence, since the action of Γ on \bar{F} is nilpotent and this has a filtration by Γ -invariant subspaces with $\bar{F}/[\Gamma, \bar{F}] \simeq \mathbb{Z}/p$ the maximal quotient on which the Γ -action is trivial, we get that $\bar{\gamma}$ generates \bar{F} . But then γ must generate F because any proper submodule of F is mapped into a proper submodule of \bar{F} under the reduction map. \square

Corollary 5.7. *If M is any finite Γ -module, the image of $H^*(\Gamma, M)$ (or what amounts to the same thing, $H_s^*(\Gamma, M)$) in $H_s^2((\Omega \times T)/\Gamma, \tilde{M})$ is zero.*

Proof. Combine Proposition 5.2 and Proposition 5.6. \square

Corollary 5.8. *For any finite H_p -module N , $H_s^i(H_p, N) = 0$ for $i \geq 3$.*

Proof. We have three fibrations: first the fibration F given by $BH_p \rightarrow B\Gamma$ with typical fiber $B\mathbb{Z}/p$, secondly the fibration, call it F' ,

$$(3) \quad f : f^{-1}((\Omega \times T)/\Gamma) \rightarrow (\Omega \times T)/\Gamma,$$

and lastly, the fibration $f|_{f^{-1}(U)}$, call it F'' where $U \subset (\Omega \times T)/\Gamma$ is a Zariski open subset such that the maps from $H^i(\Gamma, F)$ to $H^i(U, F)$ factors over $H_s^i(\Gamma, F)$ for any Γ -module F resp. induced local system F on U . We get maps of fibrations $F'' \rightarrow F' \rightarrow F$, and corresponding maps of the associated Serre spectral sequences of the fibrations.

We want to show that the image of $H^i(BH_p, \tilde{N})$ in $H^i(f^{-1}(U), \tilde{N})$ is zero for all $i \geq 3$. Now this follows from an inspection of the corresponding maps of spectral sequences taking into account the following facts:

- the image of $H^2(B\Gamma, \mathcal{H}^j(B(\mathbb{Z}/p), \tilde{N}))$ in $H^2(U, \mathcal{H}^j(\mathbb{C}^*, \tilde{N}))$ is zero by Corollary 5.7.

- the image of $H^i(\mathrm{B}\Gamma, \mathcal{H}^j(\mathrm{B}(\mathbb{Z}/p), \tilde{N}))$ in $H^i(U, \mathcal{H}^j(\mathbb{C}^*, \tilde{N}))$ for $i \geq 3$ is zero by the choice of U ;
- the image of $H^i(\mathrm{B}\Gamma, \mathcal{H}^j(\mathrm{B}(\mathbb{Z}/p), \tilde{N}))$ in $H^i(U, \mathcal{H}^j(\mathbb{C}^*, \tilde{N}))$ for $j \geq 2$ is zero since then $H^j(\mathbb{C}^*, \tilde{N}) = 0$.

□

Theorem 5.9. *The stable cohomological dimension of the Heisenberg group H_p is equal to 2. In particular, the bound given in Theorem 3.1 (equal to 3 in the present case) is not sharp.*

Proof. By Corollary 5.8, the stable cohomological dimension of H_p is ≤ 2 and it suffices to show that

$$H_s^2(H_p, \mathbb{Z}/p) \neq 0$$

(where \mathbb{Z}/p are trivial/constant coefficients). We use the computation of the usual group cohomology $H^2(H_p, \mathbb{Z}/p)$ in [Leary]: in particular, it is shown there (Theorem 6 resp. Theorem 7, cf. also the reasoning p. 70, l. 5 ff.) that there exists an element $Y \in H^2(H_p, \mathbb{Z}/p)$ restricting to a product of elements of degree 1 on a subgroup $\mathbb{Z}/p \oplus \mathbb{Z}/p$ in H_p . Such a Y must be stable. □

We want to conclude with the following

Conjecture 5.10. Let G be a finite p -group. Let d be the stable cohomological dimension of G . Then there exists a G -module A with trivial action such that $H_s^d(G, A) \neq 0$.

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